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U.S. AIR FORCE
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DERIVATION OF TWO SIMPLE METHODS FOR THE
COMPUTING OF RADIOACTIVE FALLOUT

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RM-2460

February 18, 1960

Assigned to_____

SUMMARY

This report presents the mathematical derivation of two methods for computing the radioactive fallout at a point. Information is given which will permit the final calculation of dose or dose rate.

CONTENTS

I. INTRODUCTION

It is imperative, in the nuclear age, that we be able to evaluate the effects of radiological fallout. In order to obtain an adequate representation of the fallout field which may result from an atomic campaign, it is necessary first to understand the mechanism of fallout; second, to have a simple realistic method for estimating the fallout field from a single bomb; and, third, to have a simple and realistic method for combining the patterns from many bombs into a comprehensive picture of the distribution and intensity of fallout.

The basic mechanism of fallout has been understood for some time.[1,2] However, the development of a simple scheme for estimating the fallout from a single bomb has been delayed because of the uncertainties of the distribution of activity within the cloud formed by the detonation, the distribution of activity with particle size, the fall rate of the particles, and the determination of the true velocity of the wind. A recent study, reported in a series of papers,[3-10] addressed the problems of estimating the necessary parameters and determining the effect of changes in these parameters on the final fallout patterns. The purpose of this paper is to present a brief outline of the fallout mechanism, to derive from this mechanism a mathematical model to represent the main features of the fallout process, and to provide a detailed exposition of the simple solutions to this mathematical model.

It is expected that, having derived these solutions of mathematical models which can adequately describe fallout patterns over a fairly wide range of yield and wind conditions, the next step can be taken: namely,

the development of easy methods for applying the solutions. It has long
been the aim of those working on the theory of fallout to develop a
relatively streamlined method for computing fallout, and it is believed that
the development described here will lead to such a tool.

II. BASIC FACTORS

The detonation of a nuclear device on or above the ground creates a
hot bubble of gas containing all of the fission products of the detonation.
This hot bubble will rise and, because of the strong temperature gradient,
form a vortex ring. The vortex ring, by virtue of its stability, will
trap the fission fragments and carry them aloft into the mushroom part of
the cloud. If the detonation is on or very close to the surface of the
earth, large amounts of dirt will be swept into the circulation and the
fission fragments will be deposited on particles of various sizes. If
the detonation is so high in the air that the vortex ring is formed before
appreciable quantities of earth are swept into it, the particles which are
formed will be extremely small and will not fall during the first few hours.
The distribution of the activity with particle size has a direct bearing
on the close—in fallout. If there is considerable earth swept into the
cloud, approximately 80 per cent of the fission fragments, and also any
induced radioactivity, will be lodged on particles which will fall to the
ground within 24 hours.

III. THE MODELS

This paper will deal only with weapons in the megaton range which are detonated on the surface of the earth. This choice of problem limitation is made because it represents the most interesting case for close—in fallout. Smaller yields or bursts at higher elevations will produce smaller and less intense patterns.

The conditions of the cloud from 5 to 7 min after the burst provide the starting point for the fallout calculations. At this time the cloud has stopped rising, the vortex ring circulation has slowed considerably, and the cloud has cooled. The fission fragments, induced activity, and earth have formed radioactive particles which are distributed throughout the cloud. For purposes of calculation, the cloud can be described by the distribution of activity as a function of position in the cloud and the size of particles on which the activity is lodged. The subsequent history of these particles is determined by the fall velocity and the wind.

If the winds are known as a function of time and space, and the rate of fall of the particles can be determined, then the final ground position of a particle of radius r, and initial height, H, can be determined. This can be expressed mathematically. If $\mathcal{D}(r,H)$ is the horizontal distance which the particle travels and $\mathcal{V}(x,y,h,t)$ is the wind velocity,

$$\mathcal{D}(r,H) = \int_{o}^{T} \mathcal{V}(x,y,h,t)\, dt$$

where the integration follows the path of the particle. The vertical velocity of the particle is given by

$$W(r,H) = \frac{dh}{dt}$$

so that the horizontal distance traveled is

$$\mathcal{D}(r,H) = \int_0^H \frac{V}{W} \, dh \tag{1}$$

Even if it were possible to determine the exact winds and fall velocities, the complete mathematical expression would make Eq. (1), which we will call the transport equation, impossible to solve. The transport equation will therefore be approximated by simple functions which are in accord with our state of knowledge of the wind and fall velocities.

The "transport equation" can, in principle, be inverted to determine the initial position and size of the particles which will land at a given point. If the activity is integrated over all such particles, the radiation from this point can be determined. Since the initial cloud condition is expressed as a density function, and since the radioactivity is not affected by the fall and transport of the particles, it is possible to deal only with the fraction of activity and time of arrival. The initial distribution of activity in the cloud may be expressed as $A(r,H,R)$, where r is the radius of the particles, H is the height above the ground, and R is the radial dimension of the cloud. If circular symmetry is assumed, $A(r,H,R)$ is the fraction of activity per micron-kft-sq n mi.

Now, the radioactivity which will affect people is a function of the density of material on the surface; therefore, if the activity function is integrated over the particle sizes, from all heights, which land at a point, the result will be the density function per sq mi.

$$F(x,y) = \int_{H_1}^{H_2} \int_{r_1}^{r_2} A(r,H,R) \, dr \, dH \tag{2}$$

This will be termed the "fraction equation" and is the second necessary equation for the mathematical model of fallout.

In order to make these two equations tractable it is necessary to make some simplifying assumptions. The assumptions will be listed in two groups, without any detailed rationale. The first group of assumptions are necessary to provide any type of solution and the second group are needed to provide practical computing procedures.

The first assumption is that the particles fall with the terminal velocity of spheres. The experimental work reported by Rapp and Sartor[11] is the principal justification for this assumption. The next assumption is that the winds, as measured by normal meteorological sounding methods, provide sufficient detail to estimate the displacement of the falling particles. Essentially, this assumes that turbulence will have no effect on the particles. This problem is discussed by Rapp.[10] With these two restrictions the transport equation has been solved numerically for space- and time-varying winds.[6]

In order to solve the fraction equation the distribution function, $A(r,H,R)$, is needed. This was assumed to be the product of three independent distributions

$$A(r,H,R) = A_1(r) \, A_2(H) \, A_3(R)$$

$A_2(H)$ and $A_3(R)$ were determined empirically from observations of the distribution of activity in a cloud at 7 min.[6] $A_1(r)$ was assumed to be a log normal distribution and the parameters of this distribution were chosen to fit the fallout pattern of a test shot, subject to the assumption that $A_2(H)$, $A_3(R)$, $V(x,y,h,t)$, and $W(r,H)$ were all known.[7] The limits of integration were then determined from the transport equation. This system of equations was solved numerically for the wind data from a nuclear test

which was not used in deriving the model and the results were found to be adequate.

It is not claimed that the model presented here is unique or precise. The model is, however, based on accepted physical principles, is consistent with meteorological practice, is comparable with model experiments, and is not inconsistent with the results of nuclear test measurements. The solution to the problem of fallout computation based on the restrictions outlined above was possible only by a lengthy numerical integration using high-speed computing machinery. In order to provide a quicker and more simple scheme for making fallout computations some further assumptions are needed.

If a mean value theorem is applied to the velocity term in the transport equation, then

$$\mathcal{D}(r,H) = \overline{V} \int_0^H \frac{dh}{W(r,h)} = \overline{V} \; T(r,H) \tag{3}$$

where $T(r,H)$ is the time it takes for a particle of radius r to fall from a height, H. By numerical integration of the terminal velocities of particles from various heights, it was found that the fall times could be approximated quite well by

$$T(r,H) = \alpha(r) + \frac{\beta(r) \; H}{r^2} \tag{4}$$

The functions $\alpha(r)$ and $\beta(r)$ are parameters which vary only slowly over the range of r and therefore may be assumed constant over limited ranges of r without committing any serious error. When using the transport equation in the form of Eq. (3) the appropriate velocity should be some sort of a weighted mean velocity.

The fraction equation is further simplified by replacing the continuous empirical distributions $A_2(H)$ and $A_3(R)$ by a constant value between limits. Thus, if $A_3(R)$ is considered to be a constant, $\bar{A}_3$, for $0 < R < R_e$, where R_e is an effective radius, it can be taken outside the integral sign. Similarly $A_2(H)$ may be replaced by a constant, $\bar{A}_2$, and taken outside the integral sign. The product of $\bar{A}_2$ and $\bar{A}_3$ is simply the reciprocal volume of the cloud, $\bar{A}_4$, so that the fraction equation becomes

$$F(x,y) = \bar{A}_4 \int_{H_B}^{H_T} \int_{r_1(H)}^{r_2(H)} A_1(r) \, dr \, dH \tag{5}$$

The geometry of the situation is shown in Fig. 1. The shaded area represents a cross section of the cloud and is assumed to have a constant concentration of activity in space. The particle of radius r at height H at a distance, R, for the center will fall to the ground during time T (Eq. 4) and will travel a distance, $|\overline{V}|T$, where $\overline{V}$ is the mean wind. The height of the base, H_B, the height of the top, H_T, and the effective radius, R_e define the limits of the cloud. The coordinate axes are H, vertically upward; x, from ground zero in the direction of $\overline{V}$ and y from ground zero perpendicular to the direction of $\overline{V}$.

The direction and magnitude of $\overline{V}$ can be computed by many methods. One of the most convenient methods is to add vectorially 7/16 of the 500 mb wind vector (18,000 ft), 5/16 of the 100 mb wind vector (50,000 ft), and 4/16 of the 50 mb wind vector (68,000 ft).

There is one final bit of information needed to complete the method of calculation: the variation of the cloud parameters with yield. The Summary Report of RAND Work on the AFSWP Fallout Project [10] presents

Fig. 1 — Geometry of cloud at stabilization

some scaling laws which were based on the study of many observed clouds. These laws were the result of the fitting of least squares to the observations. Since the fitting procedure resulted in irrational numbers for the powers of the yield of the device, a consistent and simple set of scaling laws is desired. It is assumed that the effective volume of the cloud is directly proportional to the blast yield

$$Vol = C_1 W$$

where W is the yield of the device. Simple expressions for the effective radius and height difference were found to be

$$R_e = C_2 W^{0.45}$$

$$H_t - H_B = C_3 W^{0.1}$$

C_3 was found to be 22 kft, and C_2 was determined as 8.06 n mi, so that $C_1 \approx 4500$ n mi^2 kft. In addition to the scaling of the cloud dimensions the cloud height was found to scale as

$$H_B = 28.8 \ W^{0.1}$$

In all these scaling laws horizontal dimensions are measured in n mi, vertical dimensions in kft, and yield in MT.

IV. HEIGHT-PARTICLE SIZE METHOD OF INTEGRATION

The purpose of this section is to find a solution to Eq. (5), which will utilize Eq. (4), to determine the limits of integration. The problem now becomes one of determining the range of particle sizes which will reach each point downwind of ground zero. At each height, H, within the cloud, the maximum sized particle, r_2, which lands at a given point will fall from the front of the cloud and the minimum sized, r_1, from the rear of the cloud. If the cloud were infinitely thin the activity at the point would be that fraction of the activity on the particles in the range from r_1 to r_2. However, because the cloud is of finite thickness, the range of particles is a function of height and the activity at the point becomes the sum of the fractions of the activity on the particles in the range from r_1 to r_2 from each height within the cloud.

In order to integrate Eq. (5), the functions $r_1(H)$ and $r_2(H)$ must be determined. The radius of a particle, r, falling at an arbitrary point on the ground (x,y), from an initial height, H, through a constant wind field, $\overline{V}$, is uniquely determined by its time of fall, $T(r,H)$. It has been assumed that the time of fall may be approximated by Eq. (4). Let r_{1B} and r_{2B} be the radii of the smallest and largest particles falling from the base of the cloud to the point (x,y). The particles, $r_1(H)$ and $r_2(H)$, which are vertically above r_{1B} and r_{2B}, will reach the same point on the ground at the same time. Thus

$$T(r,H) = \alpha + \frac{\beta H}{r^2} = \alpha + \frac{\beta H_B}{r_{2B}^2} \tag{6a}$$

$$r_2(H) = r_{2B}\sqrt{H/H_b} \tag{6b}$$

Similarly

$$r_1(H) = r_{1B}\sqrt{H/H_B} \tag{6c}$$

Here it has been assumed that α and β remain constant over the range of particles from the bottom to the top of the cloud.

The activity-particle size distribution has been approximated by a log-normal distribution with mean, μ, and standard deviation, σ ($\mu = 3.8$ and $\sigma = 0.69$).[7] Using the log-normal distribution and Eq. (6) for the limits of integration, Eq. (5) becomes

$$F(x,y) = \frac{\bar{A}_4}{\sqrt{2\pi}} \int_{H_B}^{H_T} \int_{\ln r_{1B}\sqrt{H/HB}}^{\ln r_{2B}\sqrt{H/H_B}} e^{-1/2\left(\frac{\ln r - \mu}{\sigma}\right)^2} \, d(\ln r)\, dH \tag{7}$$

If $s = \frac{\ln r - \mu}{\sigma}$, and $ds = \frac{1}{\sigma} d(\ln r)$

then $F(x,y) = \bar{A}_4 \int_{H_B}^{H_T} \int_{Q_1}^{Q_2} e^{-1/2 s^2} \, ds\, dH$

where $Q_1 = \dfrac{\ln\left(r_{1B}\sqrt{H/H_B}\right) - \mu}{\sigma}$

$Q_2 = \dfrac{\ln\left(r_{2B}\sqrt{H/H_B}\right) - \mu}{\sigma}$

$$= \bar{A}_4 \int_{H_B}^{H_T} \left\{ \bar{\phi}\left[\frac{\ln\left(r_{2B}\sqrt{H/H_B}\right) - \mu}{\sigma}\right] - \phi\left[\frac{\ln\left(r_{1B}\sqrt{H/H_B}\right) - \mu}{\sigma}\right] dH \right\} \tag{8}$$

$$= \bar{A}_4 \left\{ P_2 - P_1 \right\}$$

where ϕ is the standard normal distribution.

Both expressions in the integrand of Eq. (8) may be treated in the same way; thus only one integration will be performed and the other written down by analogy. P_1 is given by

$$\int_{H_B}^{H_T} \phi \left[\frac{\ln r_1 \sqrt{H/H_B} - \mu}{\sigma} \right] dH$$

If $t = \dfrac{\ln (r_1 \sqrt{H/H_B}) - \mu}{\sigma}$

then

$$dt = 1/2 \; \sigma \frac{dH}{H}$$

If $H = e^{2(\sigma t - \mu - \ln r_1/\sqrt{H_B})}$,

$$t_B = \frac{\ln r_1 - \mu}{\sigma} \quad \text{and} \quad t_T = \frac{\ln (r_1 \sqrt{H/H_B}) - v}{\sigma}$$

then

$$P_1 = 2\sigma \; e^{2(\mu - \ln \frac{r_1}{\sqrt{H_B}})} \int_{t_B}^{t_T} \phi(t) \; e^{2\sigma t} \; dt$$

Integration by parts yields

$$P_1 = 2\sigma e^{2(\mu - \ln r_1/\sqrt{H_B})} \left[\phi(t) \; \frac{e^{2\sigma t}}{2\sigma} \; \middle|_{t_B}^{t_T} \right. -$$

$$2\sigma e^{2(\mu - \ln r_1/H_B)} \int_{E_B}^{E_T} \frac{1}{r_2 \; \sigma} \; e^{2\sigma t - 1/2 \; t^2} \; dt$$

By completing the square in the exponent

$$P_i = 2ce^{2(\mu - \ln r^2/\!\!\sqrt{H_B})} \left[\phi(t) \; \frac{e^{2\sigma t}}{2\sigma} \right|_{t_B}^{t_T}$$

$$- 2\sigma e^{2(\mu - \ln r_i/\!\!\sqrt{H_B})} \frac{e^{2\sigma^2}}{2\sigma} \int_{t_B}^{t_T} e^{-1/2(t-2\sigma)^2} \; dt$$

If $w = t - 2\sigma$, then $dt = dw$

and

$$P_i = \left\{ \phi\left(\frac{\ln r_i\sqrt{\frac{H_T}{H_B}} - \mu}{\sigma}\right) H_T - \phi\left(\frac{\ln r_i - \mu}{\sigma}\right) H_B \quad \right\}$$

$$- \left\{ \phi\left(\frac{\ln r_i\sqrt{\frac{H_T}{H_B}} - \mu - 2\sigma^2}{\sigma}\right) - \phi\left(\frac{\ln r_i - \mu - 2\sigma^2}{\sigma}\right) \right\} \frac{H_B}{r_i^2} e^{2(\mu + \sigma^2)} \quad (9)$$

An inspection of P_i shows that as $r_i \to \infty$ $P_i \to H_T - H_B$. Therefore $P_i/\Delta H \to 1$ as $r_i \to \infty$. The function $P_i/\Delta H$ was computed for several values of r_i and plotted on log-probability paper. The results indicated that

$$\psi(r_i) = \frac{P_i}{\Delta H} \cong \phi\left(\frac{\ln r_i - \mu_1}{\sigma_1}\right) \quad (10)$$

With the choice of scaling laws discussed in Section III, the function $\frac{P_i}{\Delta H} = \psi(r_i)$ is independent of yield. $\psi(r_i)$ is then a log-normal distribution with mean μ_1 equal to 3.65 and $\sigma_1 = 0.69$.

This new log-normal distribution has, in effect, reduced the problem to one of computing fallout from an infinitely thin cloud. The only change in the distribution is found in the mean, which has decreased. This decrease of the mean represents an increase of activity on the particles falling from the base by an amount equal to the sum of the activity on the particles falling from all heights, thus, in effect, concentrating all the activity at the base. The equation for fraction down becomes

$$F(x,y) = \frac{1}{\pi R_e^2} \left\{ \psi(r_{2B}) - \psi(r_{1B}) \right\} \qquad (11)$$

Up to this point no mention has been made of the meteorological variable, wind velocity, with the exception that it is connected to the fraction down through the fall velocity of the particles. As was mentioned earlier the radius of each particle falling from height H determines the time down. This time determines the horizontal distance the particle will travel in a given wind velocity. In order to solve Eq. (11) one must first find r_{2B} and r_{1B}.

Figure 2 shows a cross section of the cloud in the direction of the mean wind. This figure clearly shows that the particle r_{2B} travels a horizontal distance, $x - R_e$, and the particle, r_{1B}, a horizontal distance, $x + R_e$. With a mean wind of $\overline{V}$ the times of fall of these particles would be

$$T_2 = \frac{x - R_e}{\overline{V}} \qquad (12)$$

and

$$T_1 = \frac{x + R_e}{\overline{V}} \qquad (13)$$

This gives the times of fall for the largest and smallest particles on the
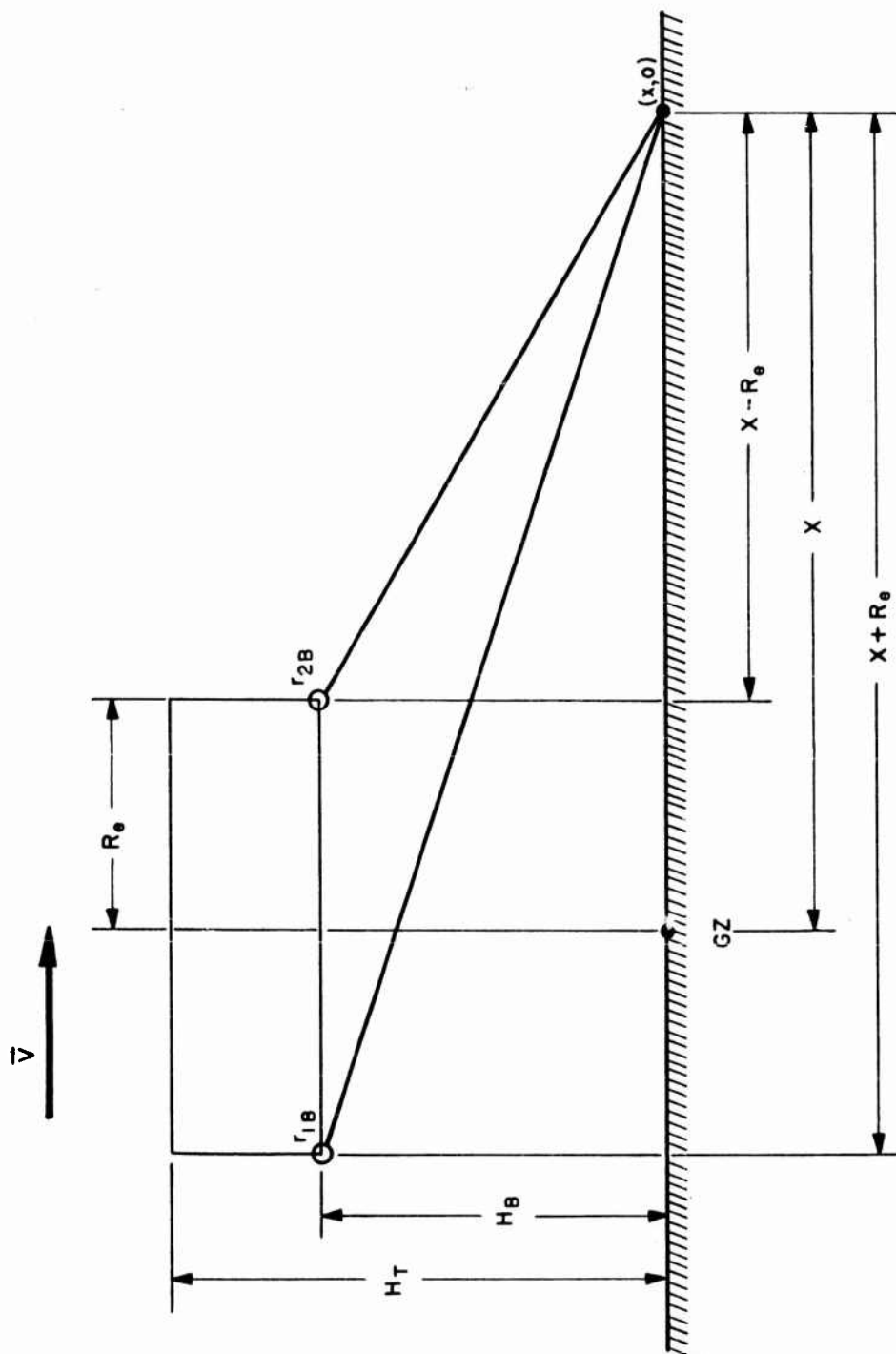
Fig. 2 — View of cloud in the X–Z plane

x-axis. Figure 3 shows the cloud from the top. It can be seen that particles falling from the cloud at a distance, y, from the x-axis will travel different horizontal distances. These distances depend upon the effective radius of the cloud at y. By geometry the effective radius becomes

$$R_e' = R_e \sqrt{1 - (y/R_e)^2}$$

Thus, for off-axis points, Eqs. (12) and (13) become

$$T_2 = \frac{x - R_e'}{\bar{V}} = \frac{x - R_e f(y)}{\bar{V}} \tag{12a}$$

and

$$T_1 = \frac{x + R_e'}{\bar{V}} = \frac{x + R_e f(y)}{\bar{V}} \tag{13a}$$

Here $f(y)$ is introduced to scale the cloud radius for the off-axis points. By geometry $f(y)$ is

$$f(y) = \sqrt{1 - \left(\frac{y}{R_e}\right)^2} \tag{14}$$

In theory, r_{1B} and r_{2B} can be computed for various yields from Eq. (4) and the scaling law for H_B. Because of the dependence of α and β on r it is convenient to graph r as a function of T and W (see Fig. 4).

This method may also be used to compute the fraction down at points under the cloud and behind ground zero. The range of particle sizes falling from the base to a point under the cloud will include all particles larger than the size of those falling from the rear of the cloud. The function $\psi(r_{2B})$ will then become unity and the fraction down equation becomes
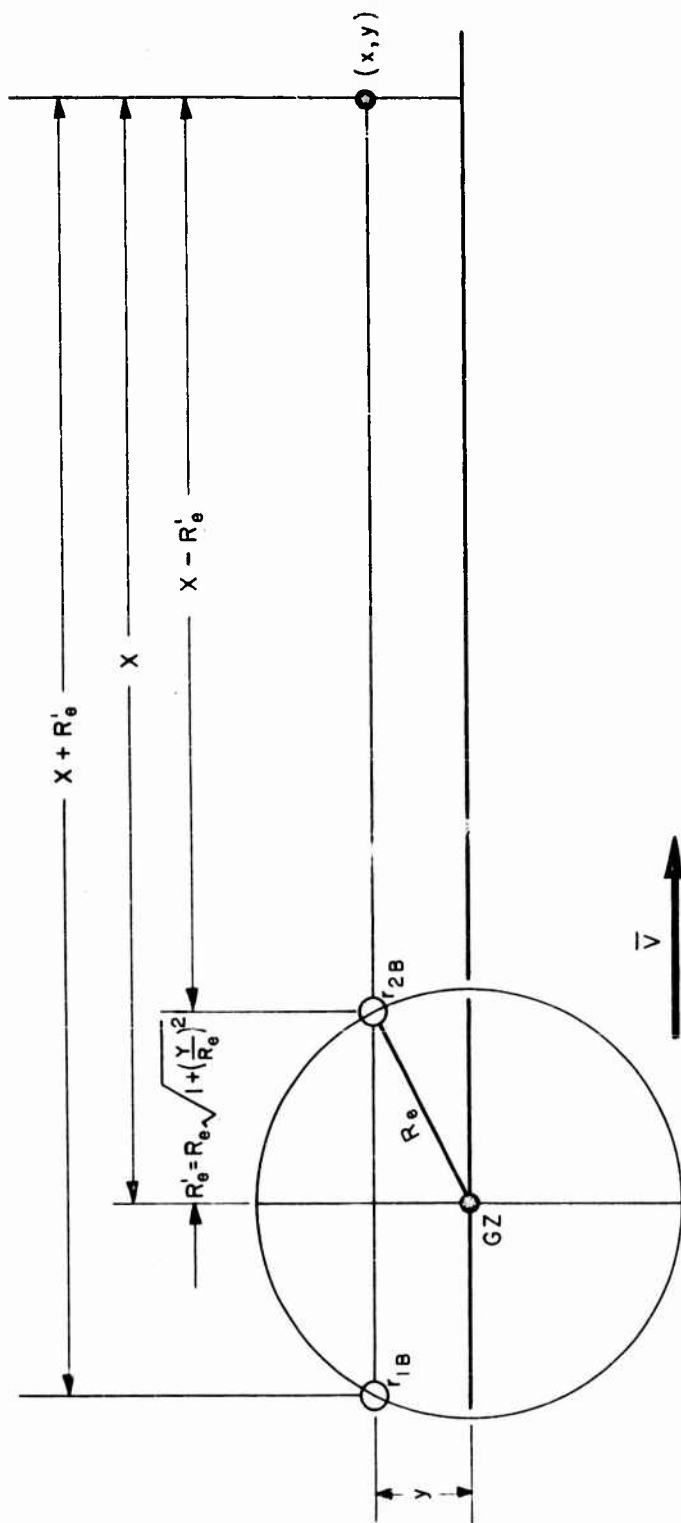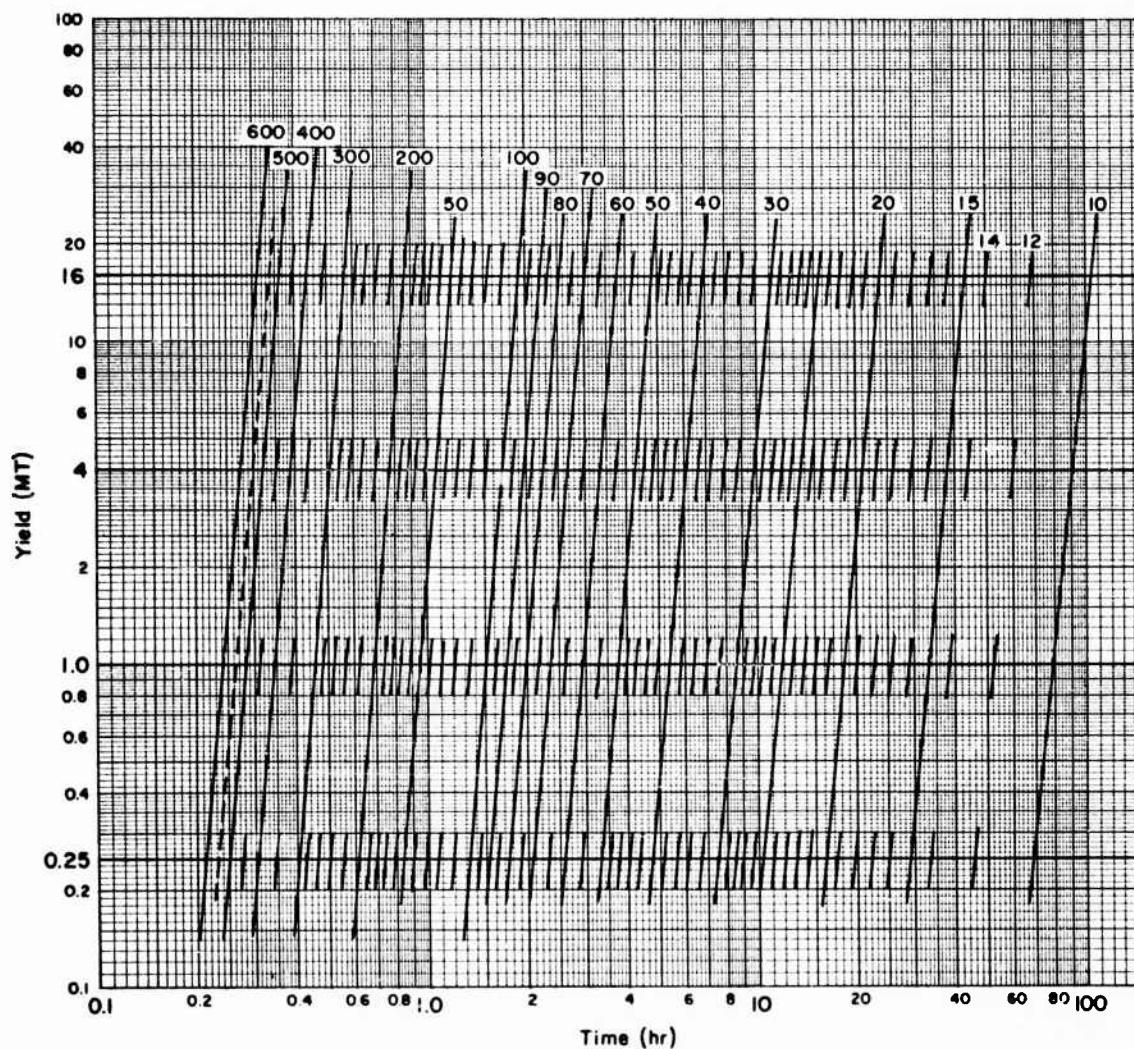
Fig. 3 — View of cloud in the X-Y plane

Fig. 4—Particle size as a function of yield and time

$$F(x,y) = \frac{1}{R_e^2} \left[1 - \psi(r_{1B}) \right] \tag{15}$$

The fraction down is obviously a function of yield and wind velocity. It might be helpful at this point to describe how the fraction changes with changing yield and wind velocity. From Eq. (11) it can easily be seen that $F(x,y)$ varies inversely with the square of the cloud radius. Thus, as the yield increases, the fraction decreases.

Another change in the fraction can be attributed to the cloud radius. It is clear from Eqs. (12) and (13) that, for a given $\bar{V}$, the period during which particles arrive at a point increases with increasing cloud radius. The resulting increase in the range of particle sizes will increase the fraction down. In connection with Eqs. (12) and (13), it can be seen that, for a given (x,y), an increase of wind velocity would produce a decrease in the range of particle sizes and a shift of that range to larger particle sizes. The last effect is attributed to the increasing H_B with increasing yield. For a given (x,y) an increase in yield produces a shift in the range of particles toward the large end of the spectrum. All these effects working together make any method of scaling the fraction for yield too complicated to be useful.

In the next few paragraphs a step–by–step procedure for computing the fallout at a point will be outlined.

The quantities given for fallout patterns are yield and wind velocity. As an example, consider a 4–MT device and a mean wind of 10 kn. The procedure is as follows for points on the x axis:

A. Determine the cloud radius and $1/\pi R_e^2$ from Fig. 5

$$R_e = 15 \text{ n mi}$$
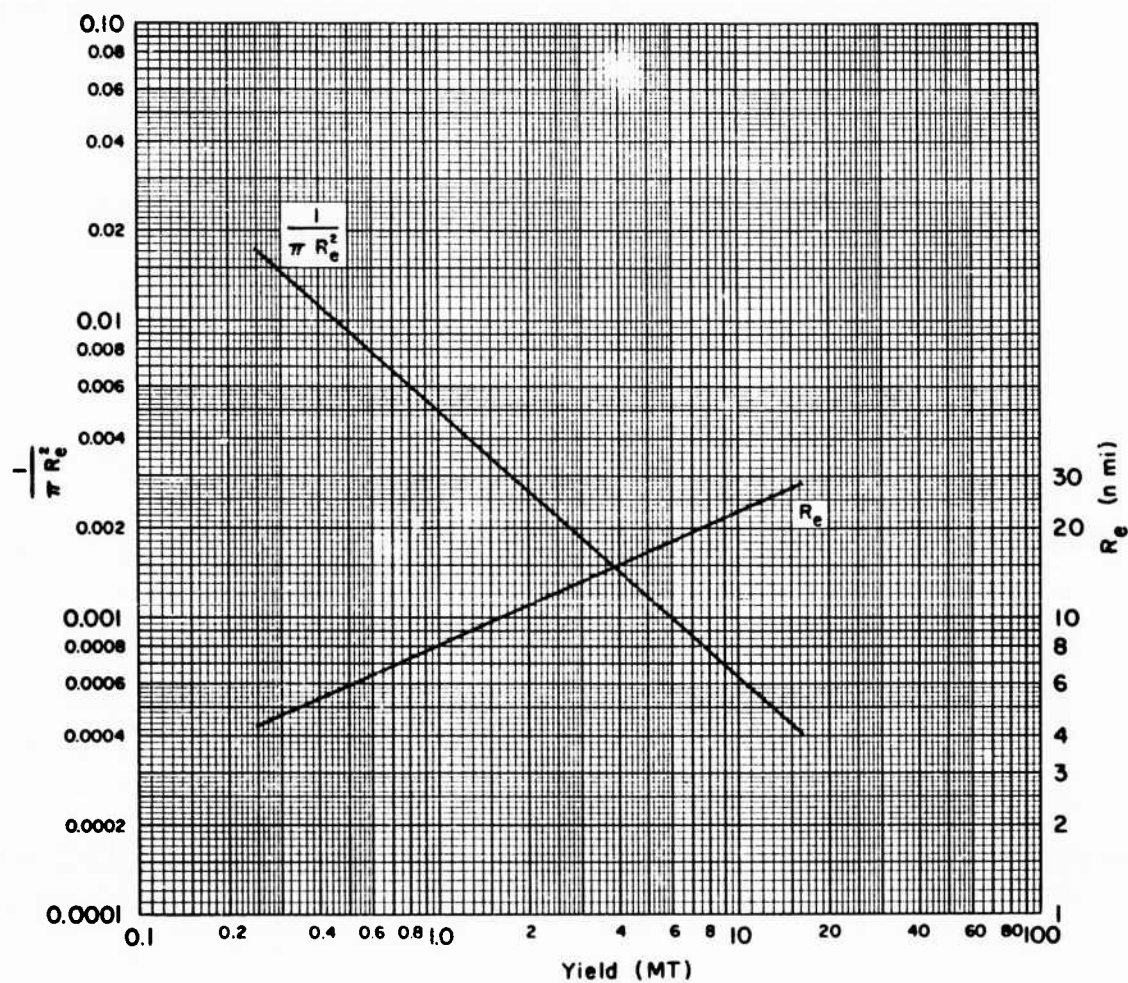
$$1/\pi R_e^2 = 0.00141$$

Fig. 5 — Cloud radius (R_e) and the reciprocal of cloud area $\left(\frac{1}{\pi R_e^2}\right)$ as a function of yield

B. Determine the downwind distances desired and record.

C. Compute T_2 and T_1 from Eqs. (12) and (13) and record.

D. Determine the particle sizes for the appropriate yield and times from Fig. 4 and record.

E. Determine ψ (r_{1B}) and $\psi(r_{2B})$ from Fig. 6 and record.

F. Take the difference of ψ (r_{1B}) and ψ (r_{2B}), then multiply by $1/\pi R_e^2$. The result is $F(x,y)$.

For points off the x-axis the same procedure is followed with the exception that the function $f(y)$ is used in determining the times T_2 and T_1.

For example, if $x = 50$ and $y = 1/2R$, then

$$T_2 = \frac{50 - 15\sqrt{1 - (1/2)^2}}{10} = 3.7 \text{ hr}$$

$$T_1 = \frac{50 + 15\sqrt{1 - (1/2)^2}}{10} = 6.3 \text{ hr}$$

Using these times, the following values of particle size are obtained
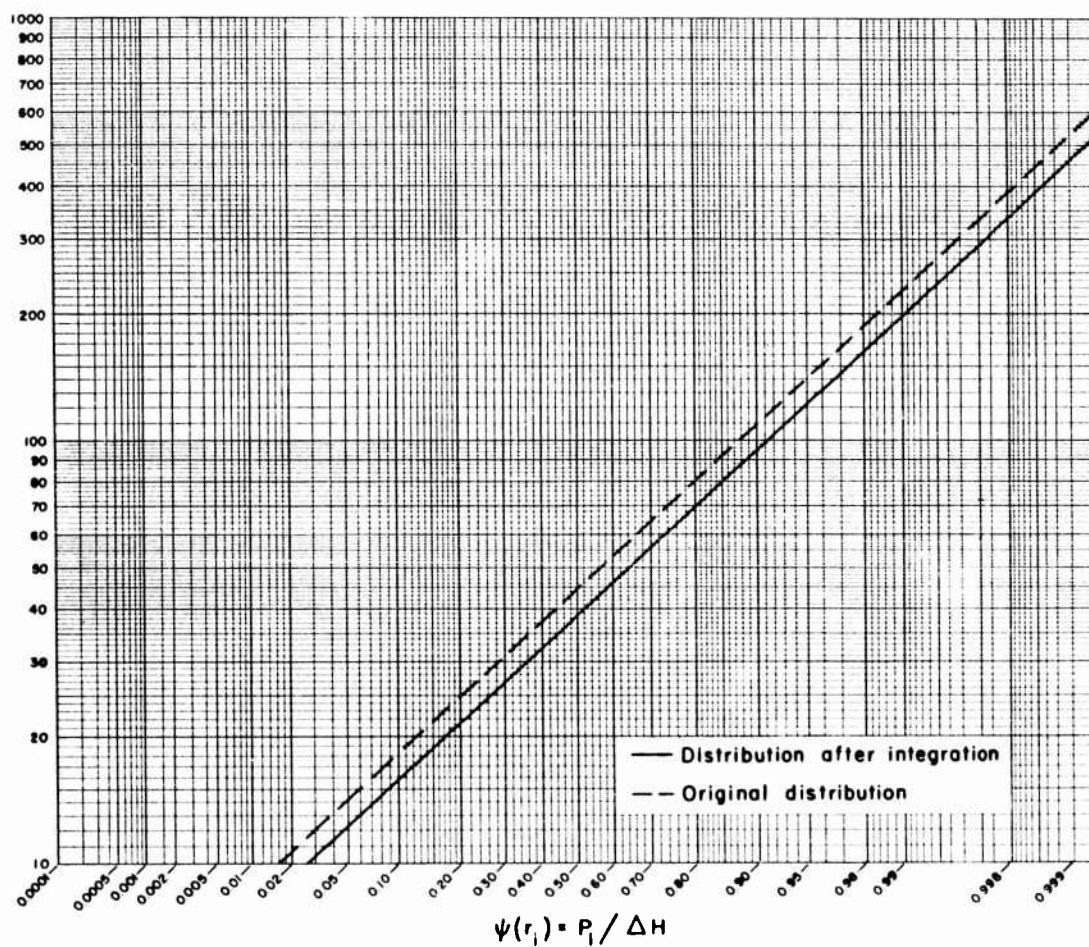
$$r_{2B} = 55$$
$$r_{1B} = 39$$

$$\psi(r_i) = P_i / \Delta H$$

Fig. 6 — The log normal distribution before and after integration

V. TIME-PARTICLE SIZE METHOD OF INTEGRATION

We proceed to perform the integration of the fraction equation to obtain
the fraction of activity per unit area, using $\overline{A}_4 A_1(r)$ to express the distri-
bution of activity in the cloud. Let us first consider calculating the
fraction for a point on the positive x-axis at a distance, $x > R$. If we
perform the integration in the r-H plane indicated in Eq. (5), we wish to
integrate over the area indicated in Fig. 7. Substantial simplification in
the final expression for the fraction seems possible if we consider a
transformation from the H-r plane into the t-r plane, where t is the time
from stabilization of the cloud (see Fig. 8).

The relation between T, H, and r is given by Eq. (4), where for the
moment we consider $\alpha(r)$ and $\beta(r)$ to be constants. Thus our transformations
are

$$T = \alpha + \frac{\beta H}{r^2}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (16)

$$r = r$$

The region over which we desire to integrate becomes that shown in Fig. 7.
Our element of area is transformed by

$$drdH = \frac{\partial(H,r)}{\partial(T,r)}\ dtdr \qquad\qquad\qquad (17)$$

where $\dfrac{\partial(H,r)}{\partial(T,r)}$ is the Jacobian of (H,r) with respect to (t,r).

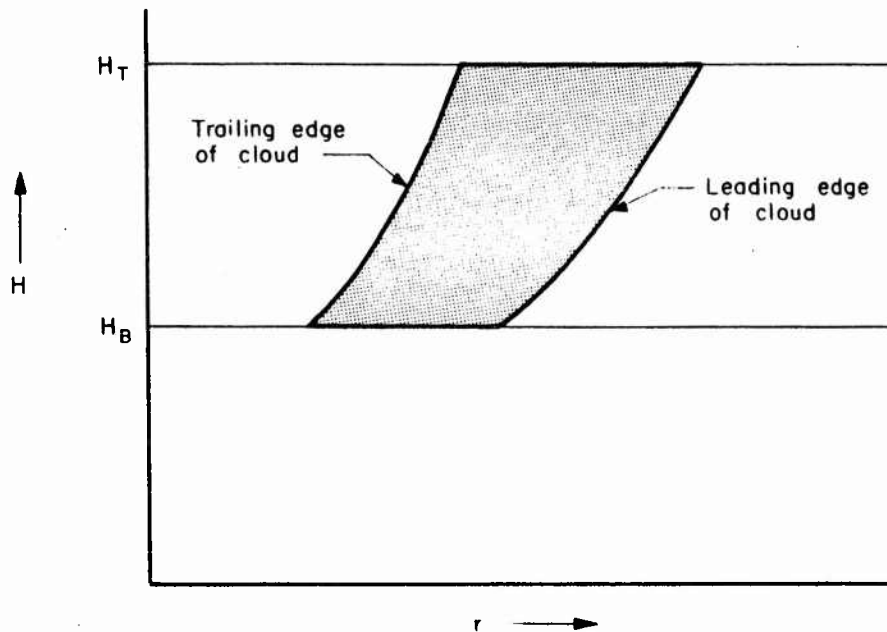$$\frac{\partial(H,r)}{\partial(T,r)} = \frac{r^2}{\beta} \qquad\qquad\qquad (18)$$

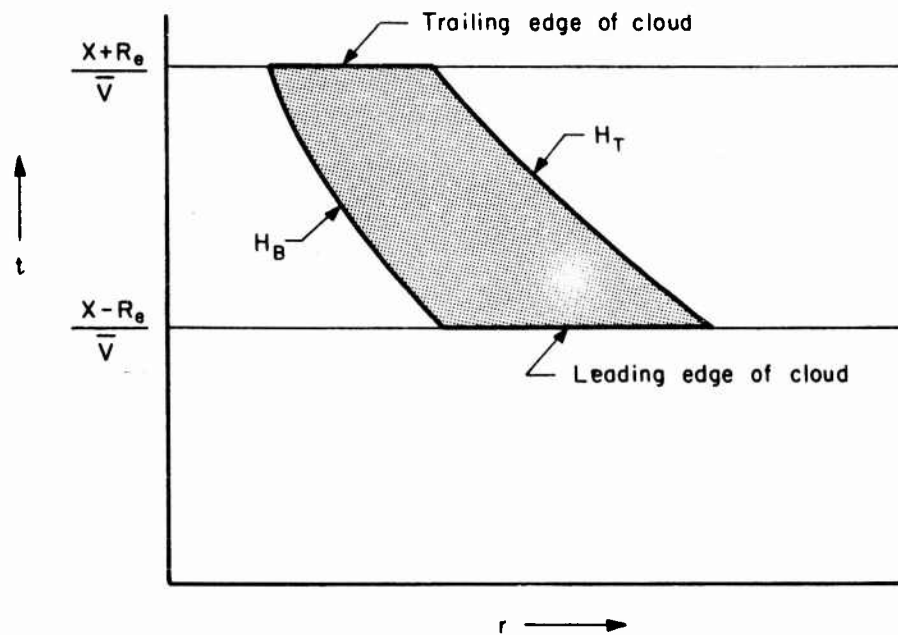Fig. 7 — Schematic diagram of the region of integration in the r-H plane

Fig. 8 — Schematic diagram of the region of integration in the r-t plane

The integration of Eq. (5) becomes

$$F(x,y) = \bar{A}_4 \int_{T_1}^{T_2} \int_{r_1(T)}^{r_2(T)} A_1(r) \frac{r^2}{\beta} \, dr \, dT \tag{19}$$

where

$$r_1(T) = \sqrt{\frac{\beta H_B}{T - \alpha}}$$

$$r_2(T) = \sqrt{\frac{\beta H_T}{T - \alpha}} \tag{20}$$

and

$$T_2 = \frac{X + R_e}{\bar{V}}$$

$$T_1 = \frac{X - R_e}{\bar{V}} \tag{21}$$

Inserting the log-normal distribution in Eq. (19) for $A_1(r)$, we obtain

$$F(x,y) = \bar{A}_4 \int_{T_1}^{T_2} \int_{\frac{\beta H_B}{T-\alpha}}^{\frac{\beta H_T}{T-\alpha}} \frac{e^{-1/2\left(\frac{\ln r - \mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma\, r} \, \frac{r^2}{\beta} \, dr \, dT \tag{22}$$

Making the substitution

$$\eta = \frac{\ln r - \mu}{\sigma} \tag{23}$$

Eq. (22) becomes

$$F(x,y) = \frac{\bar{A}_4}{\sqrt{2\pi}\,\beta} \int_{T_1}^{T_2} \int_{n}^{m} e^{-1/2\,\eta^2 + 2(\eta\sigma+\mu)} \, d\eta \, dT \qquad (24)$$

where

$$m = \frac{\ln\sqrt{\dfrac{\beta H_T}{T-\alpha}} - \mu}{\sigma}$$

and

$$n = \frac{\ln\sqrt{\dfrac{\beta H_B}{T-\alpha}} - \mu}{\sigma} \qquad (25)$$

By completing the square in the exponent, Eq. (23) becomes

$$F(x,y) = \frac{\bar{A}_4\, e^{2\mu+2\sigma^2}}{\beta} \int_{T_1}^{T_2} \int_{n}^{m} \frac{e^{-1/2(\eta-2\sigma)^2}}{\sqrt{2\pi}} \, d\eta \, dT \qquad (26)$$

The cumulative normal distribution function, $\phi(x)$, is defined as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\eta^2}{2}} \, d\eta \qquad (27)$$

Then Eq. (26) may be written simply as

$$F(x,y) = \frac{A_4\, e^{2\mu+2\sigma^2}}{\beta} \int_{T_1}^{T_2} \left[\phi(m-2\sigma) - \phi(n-2\sigma)\right] \, dT \qquad (28)$$

To perform the integration with respect to T we resort to integration by parts. Since the integrations of $\phi(m-2\sigma)$ and $\phi(n-2\sigma)$ are functionally the same, we will perform only the integration of $\phi(m-2\sigma)$ and write down the integral of $\phi(n-2\sigma)$ by inspection. To simplify notation, let

$$m = \frac{1}{2\sigma} \left[- \ln(T - \alpha) + C_T \right] \tag{29}$$

$$n = \frac{1}{2\sigma} \left[- \ln(T - \alpha) + C_B \right] \tag{30}$$

where

$$C_T = \ln \beta \, H_T - 2\mu$$

and

$$C_B = \ln \beta \, H_B - 2\mu$$

After making the change of variables indicated in Eq. (29) we obtain

$$I = \int_{T_1}^{T_2} \phi(m-2\sigma)dT = (- 2\sigma) \, e^{C_T} \int_{m_1}^{m_2} \phi(m-2\sigma)^{-2\sigma m} \, dm \tag{31}$$

where

$$m_2 = \frac{1}{2\sigma} \left[- \ln (T_2 - \alpha) + C_T \right]$$

and

$$\tag{32}$$

$$m_1 = \frac{1}{2\sigma} \left[- \ln (T_1 - \alpha) + C_B \right]$$

Integrating by parts, Eq. (31) becomes

$$I = -2\sigma e^{C_T} \left\{ \left[\frac{\phi(m-2\sigma)e^{-2\sigma m}}{(-2\sigma)} \right]_{m_1}^{m_2} - \frac{1}{(-2\sigma)^2} \int_{m_1}^{m_2} e^{-1/2(m-2\sigma)^2 \, -2\sigma m} \, dm \right\} \tag{33}$$

Performing the integration and simplifying Eq. (33), we have

$$I = e^{C_T} \left\{ \phi(m_2-2\sigma)e^{-2\sigma m_2} - \phi(m_1-2\sigma)e^{-2\sigma m_1} - e^{-2\sigma 2} \left[\phi(m_2)-\phi(m_1) \right] \right\} \tag{34}$$

From Eq. (34) we can immediately write $F(x,y)$ as

$$
F(x,y) = \frac{\overline{A}_4 e^{2\mu+2\sigma^2}}{\beta}
\left[
\begin{array}{l}
e^{C_T}\left\{
\begin{array}{l}
\phi(m_2-2\sigma)e^{-2\sigma m_2} - \phi(m_1-2\sigma)e^{-2\sigma n_1} \\[2mm]
-e^{-2\sigma 2}\left[\phi(m_2) - \phi(m_1)\right]
\end{array}
\right\} \\[6mm]
- e^{C_B}\left\{
\begin{array}{l}
\phi(n_2-2\sigma)e^{-2\sigma n_2} - \phi(n_1-2\sigma)e^{-2\sigma n_1} \\[2mm]
-e^{-2\sigma^2}\left[\phi(n_2) - \phi(n_1)\right]
\end{array}
\right\}
\end{array}
\right]
\tag{35}
$$

Inserting the values of C_T, C_B, m_2, m_1, n_2, and n_1 in Eq. (35) yields

$$
F(x,y) = \overline{A}_4
\left[
\begin{array}{l}
\dfrac{e^{2\mu+2\sigma^2}}{\beta}
\left\{
\begin{array}{l}
(T_2-\alpha)\ \phi(m_2-2\sigma) - (T_1-\alpha)\ \phi(m_1-2\sigma) \\[2mm]
-(T_2-\alpha)\ \phi(n_2-2\sigma) + (T_1-\alpha)\ \phi(n_1-2\sigma)
\end{array}
\right\} \\[6mm]
-H_T\left\{\phi(m_2) - \phi(m_1)\right\} + H_B\left\{\phi(n_2) - \phi(n_1)\right\}
\end{array}
\right]
\tag{36}
$$

To facilitate computation of Eq. (36) we have computed the function $\psi(T)$ which is defined as

$$
\psi(T) = \frac{(T-\alpha)\ e^{2\mu+2\sigma^2}}{\beta}\left[\phi(m-2\sigma) - \phi(n-2\sigma)\right] - H_T\phi(m) + H_B\phi(n)
\tag{37}
$$

We may now write $F(x,y)$ as

$$
F(x,y) = \overline{A}_4\left[\psi(t_2) - \psi(t_1)\right]
\tag{38}
$$

$\psi(t)$ has been computed on the JOHNNIAC* using for $\phi(x)$ the approximation derived by a suitable transformation of the function given by Hastings.[12]

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*A local RAND digital computer.

Throughout the integration $\alpha(r)$ and $\beta(r)$ have been treated as constants. It was possible to plot the time of fall vs $\alpha(r)$ and $\beta(r)$ with isopleths of constant height.[8] In computing $\psi(t)$, $\alpha(r)$, and $\beta(r)$ should vary in a continuous fashion from the values on the isopleth for H_B to those on the isopleth for H_T for the particular T being considered. Since we held $\alpha(r)$ and $\beta(r)$ constant in computing $\psi(T)$, the values of $\alpha(r)$ and $\beta(r)$ corresponded in one case to those on the isopleth for H_B and in the other to those on the isopleth for H_T. The resulting curves for $\psi(T)$, using these two sets of $\alpha(r)$ and $\beta(r)$, were the same except for a slight difference at very small times. This conclusion partially justifies the simplification of holding $\alpha(r)$ and $\beta(r)$ constant during the integration to obtain $\psi(T)$.

Since

$$T_2 - T_1 = \frac{2R}{\overline{V}} \quad \text{and} \quad \frac{T_2 + T_1}{2} = \frac{x}{\overline{V}} \tag{39}$$

$F(x,y)$ may be approximated by

$$F(x,y) = \overline{A}_4 \frac{2R}{\overline{V}} \psi'\left(\frac{x}{\overline{V}}\right) \tag{40}$$

The derivative of $\psi(T)$ can be computed directly by considering the equation

$$\psi(T) + C = \int_{0}^{T\sqrt{\frac{\beta H_T}{T-\alpha}}} \int_{\sqrt{\frac{\beta H_B}{T-\alpha}}} \frac{e^{-1/2\left(\frac{\ln r - \mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma\, r} \frac{r^2}{\beta}\, dr\, d\eta \tag{41}$$

where C is a constant. Differentiating Eq. (41) with respect to T we obtain

$$\psi'(T) = \int\limits_{\sqrt{\frac{\beta H_B}{T-\alpha}}}^{\sqrt{\frac{\beta H_B}{T-\alpha}}} \frac{e^{-1/2\left(\frac{\ln r - \mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma\, r} \frac{r^2}{\beta}\, dr \tag{42}$$

Substituting Eq. (23), Eq. (42) becomes

$$\psi'(T) = \frac{e^{2\mu + 2\sigma^2}}{\beta\sqrt{2\pi}} \int\limits_{n}^{m} e^{-1/2(\eta - 2\sigma)^2}\, d\eta \tag{43}$$

or

$$\psi'(T) = \frac{e^{2\mu + 2\sigma^2}}{\beta} \left[\phi(m-2\sigma) - \phi(n-2\sigma)\right] \tag{44}$$

$\psi'(T)$ has also been computed on the JOHNNIAC for heights appropriate to yields from 1/4 to 16 MT. These curves are shown in Figs. 9–12.

Once the fraction has been calculated for a point on the x-axis, it is a simple matter to obtain the value for a point off the axis. For a point, (x,y), the effective radius of the cloud becomes $R_e f(y)$, where

$$f(y) = \sqrt{1 - \left(\frac{y}{R_e}\right)^2} \tag{45}$$

Thus Eq. (44) becomes

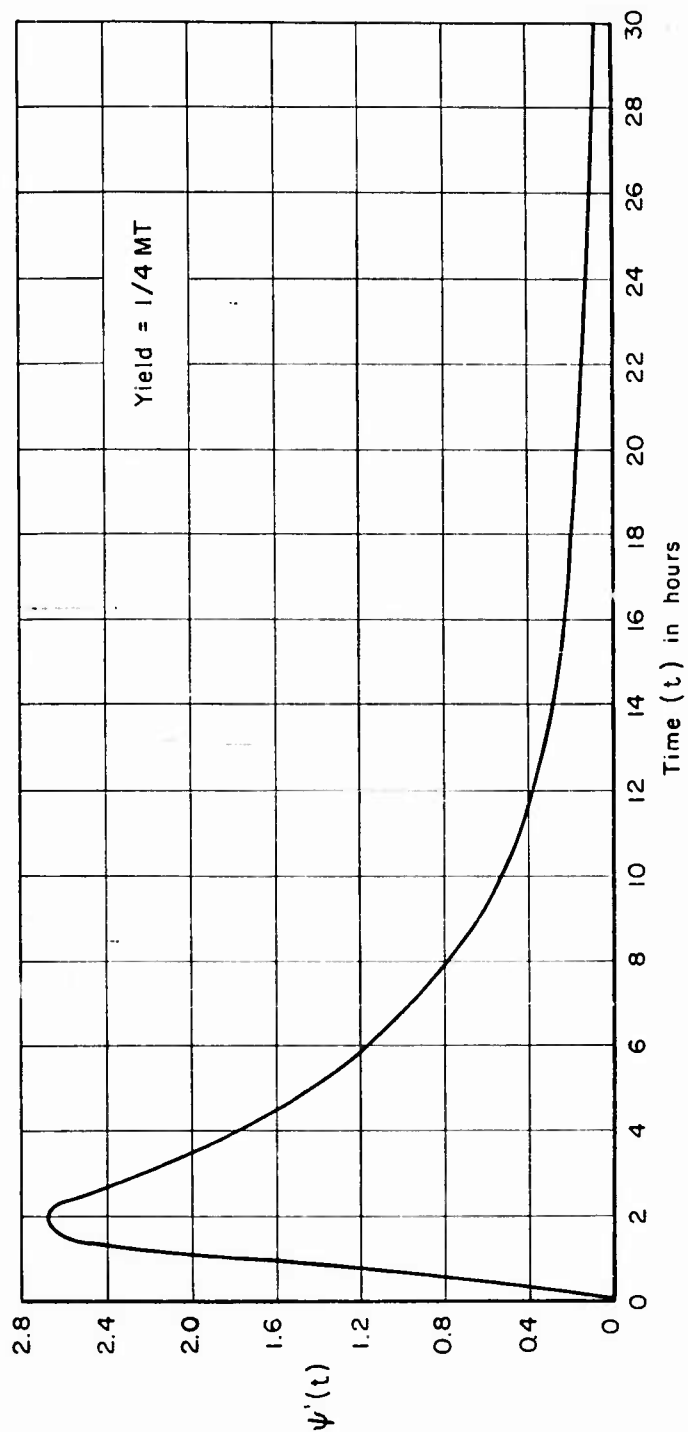$$F(x,y) = \overline{A}_4 \frac{2R}{\overline{v}} f(y) \psi'\left(\frac{x}{\overline{v}}\right) \tag{46}$$

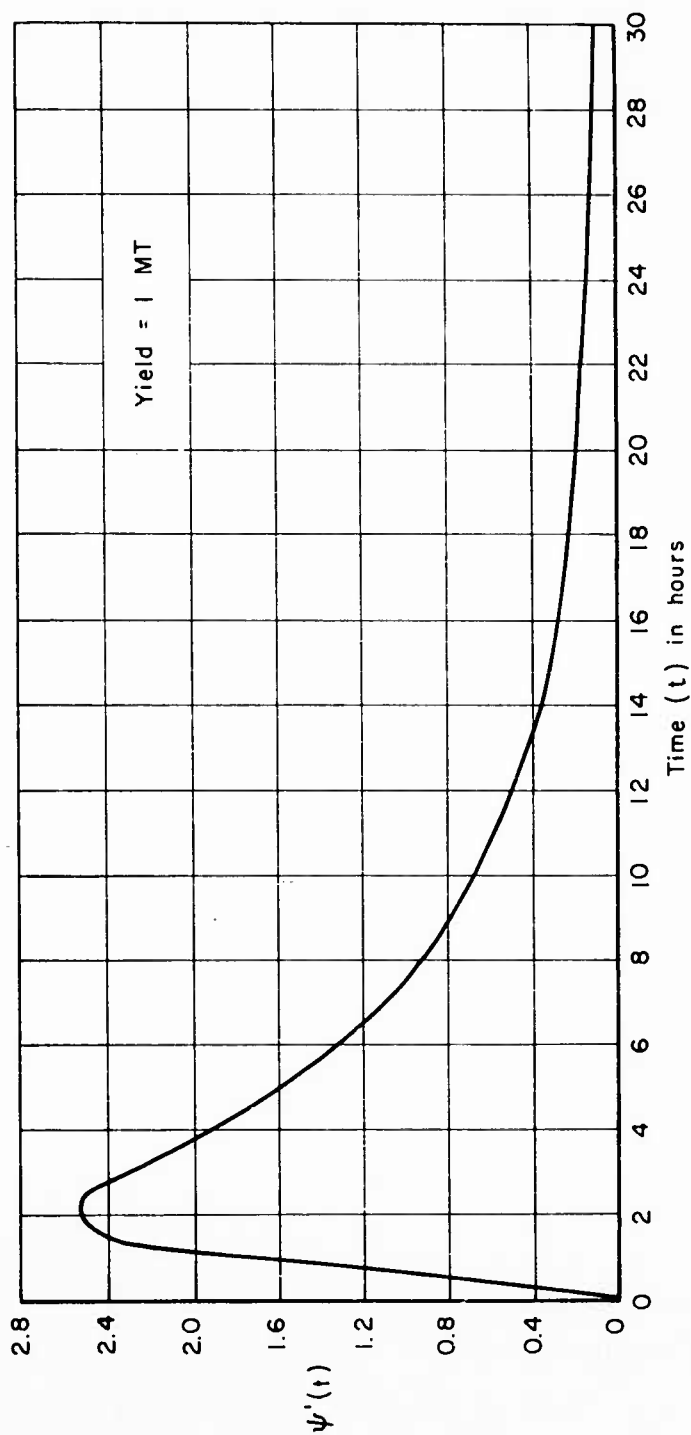Fig. 9 — Fraction of device arriving at the surface per unit time for a 1/4 MT device

Fig. 10 — Fraction of device arriving at the surface per unit time for a 1 MT device
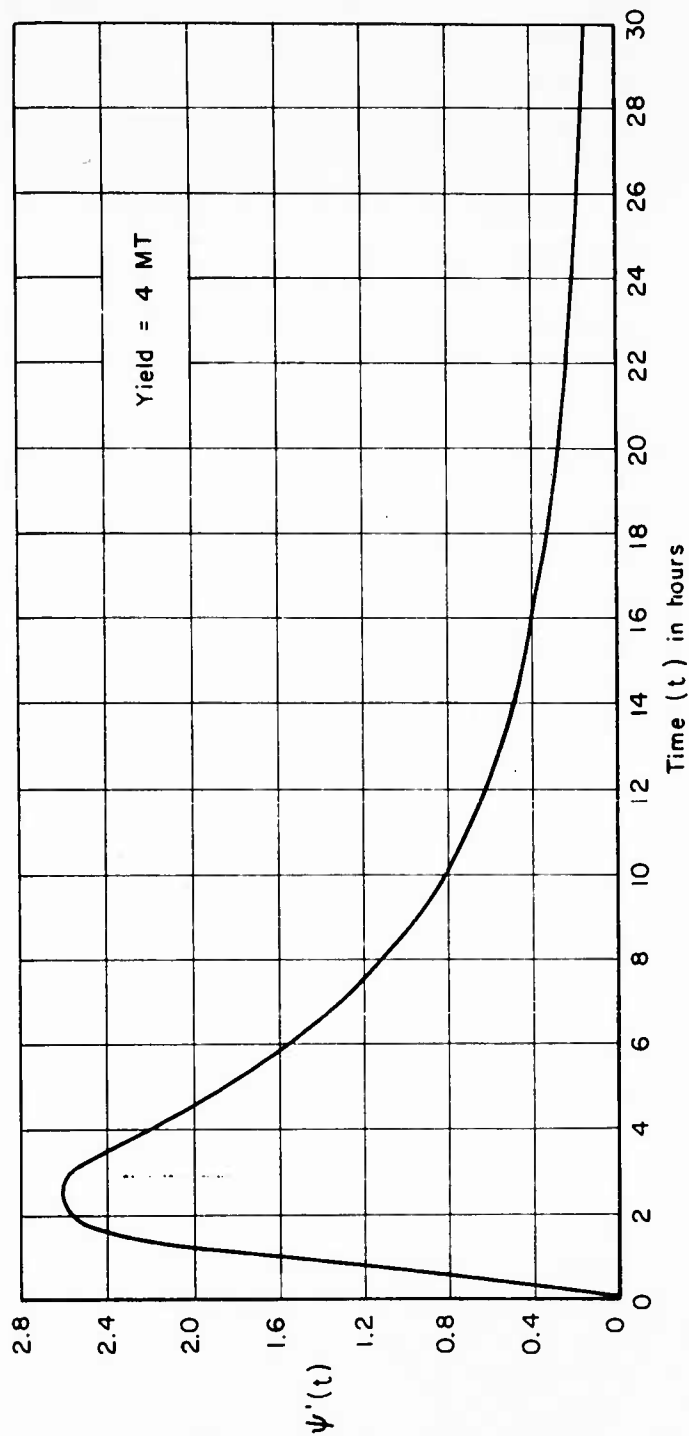
Fig. 11 — Fraction of device arriving at the surface
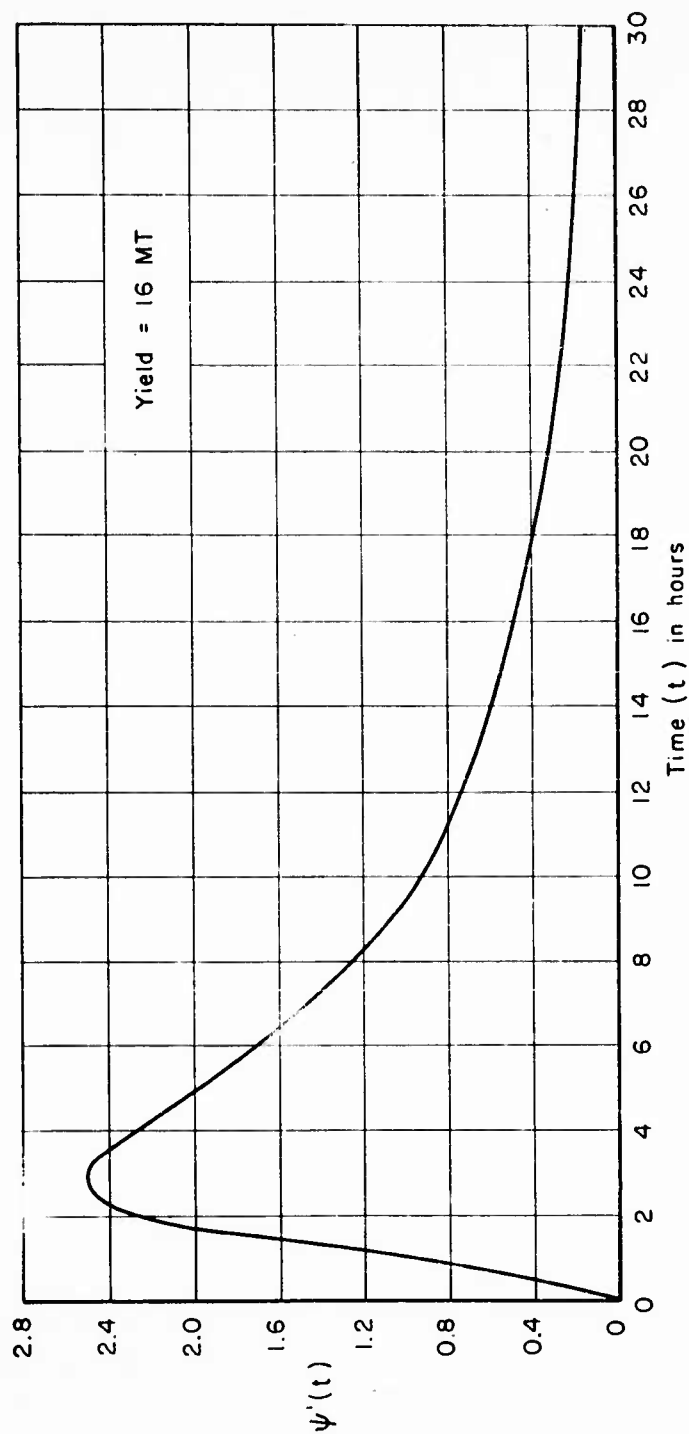per unit time for a 4 MT device

Fig. 12 — Fraction of device arriving at the surface per unit time for a 16 MT device

For points under the cloud we replace Eq. (46) by

$$F(x,y) = \begin{cases} \bar{A}_4 \dfrac{\left[R_e f(y) + x\right]}{\bar{V}} \psi' \left[\dfrac{R_e f(y) + x}{2\bar{V}}\right] & -R_e f(y) < x < R_e f(y) \\[3em] 0 & x \leq -R_e f(y) \end{cases} \tag{47}$$

As an example of the use of this technique, let us compute the fraction of activity per sq n mi at a point. For points with an x—coordinate greater than R_e we apply Eq. (46). The steps in the computation are as follows.

A. Compute $\dfrac{x}{\bar{V}}$ and read the value of ψ' $(x/\bar{V})$ from the curve of $\psi'(T)$ vs T for the appropriate yield.

B. Compute $\bar{A}_4$ using the relation

$$\bar{A}_4 = \frac{1}{4500 \ W}$$

C. Read R_e from Fig. 6 for the appropriate yield and compute $\dfrac{2R_e}{\bar{V}}$.

D. Compute $f(y)$ from Eq. (45).

E. Multiply the results of A, B, C, and D to obtain the fraction of activity per sq n mi at the point (x,y).

Consider a 4—MT weapon and a mean wind of 10 kn. To compute the fraction at the point $x = 60$ n mi and $y = 5$ n mi we proceed as above.

A. $\dfrac{x}{\bar{V}} = \dfrac{60}{10} = 6$ hr , $\psi'(x/\bar{V}) = \psi'(6) = 1.57$

B. $\bar{A}_4 = \dfrac{1}{(4500)(4)} = 5.55 \times 10^{-5}$

C. $\dfrac{2R_e}{\bar{V}} = \dfrac{(2)(15)}{10} = 3$

D. $f(y) = \sqrt{1 - (5/15)^2} = .943$

E. $F(x,y) = 5.55 \times 10^{-5} \times 3 \times .943 \times 1.57 = 2.46 \times 10^{-4}$ per $(\text{n mi})^2$

To compute the fraction at points with an x-coordinate less than R_e we proceed in a manner similar to that above but we use Eq. (47) in place of Eq. (46).

VI.   DOSE RATE AND INTEGRATED DOSE

Given the fraction down at a point and its mean arrival time it is possible to estimate the dose rate at any time and the integrated dose to any time. Assume that 3×10^5 gamma Mc are produced per MT of fission yield at one hour and that the hypothetical dose rate at one hour is equal to 3 r/h per gamma Mc per sq n mi. The total number of Mc at one hour produced by a weapon of W MT, where f, the fraction of the yield due to fission, is $3Wf \times 10^5$. Hence $3Wf \times 10^5$ is the number of Mc per sq n mi and the hypothetical one hour dose rate is given by

$$R_1^{\bullet} = 9WfF \times 10^5 \ r/h \tag{48}$$

The dose rate at some later time is $R_1^{\bullet} t^{-1.2}$ provided t is larger than the time at which the fallout at this point is complete. To compute the integrated dose at a point (x,y) we make the assumption that all of the activity arrives at this point at a time equal to $\frac{x}{\bar{V}}$. Hence the integrated dose, $R(T)$, at a time, T, is given by

$$R(T) = R_1^{\bullet} \int_{\frac{x}{\bar{V}}}^{T} t^{-1.2} \, dt \tag{49}$$

or

$$R(T) = 5R_1^{\bullet} \left[\left(\frac{x}{\bar{V}} \right)^{-0.2} - T^{-0.2} \right] \tag{50}$$

For convenience of computation

$$5 \left[\left(\frac{x}{\overline{V}} \right)^{-0.2} - T^{-0.2} \right] \tag{51}$$

is plotted on Fig. 13 for the times of usual interest.

It should be pointed cut that this integration replaces the true time of arrival, which is spread over an increment in time, with an arrival time appropriate to the arrival of the particle from the center of the base of the cloud. Thus the dose rate will build up during the period of arrival and fall off after the rate of arrival becomes smaller than the rate of decay as shown in Fig. 14. The arrival time used in these calculations is somewhere near the midpoint in time between the time of first arrival, t_1, and the time of cessation, t_2. The true integrated dose is the integral under the solid curve; the integrated dose computed by the above method is the hatched area. The differences are minor.
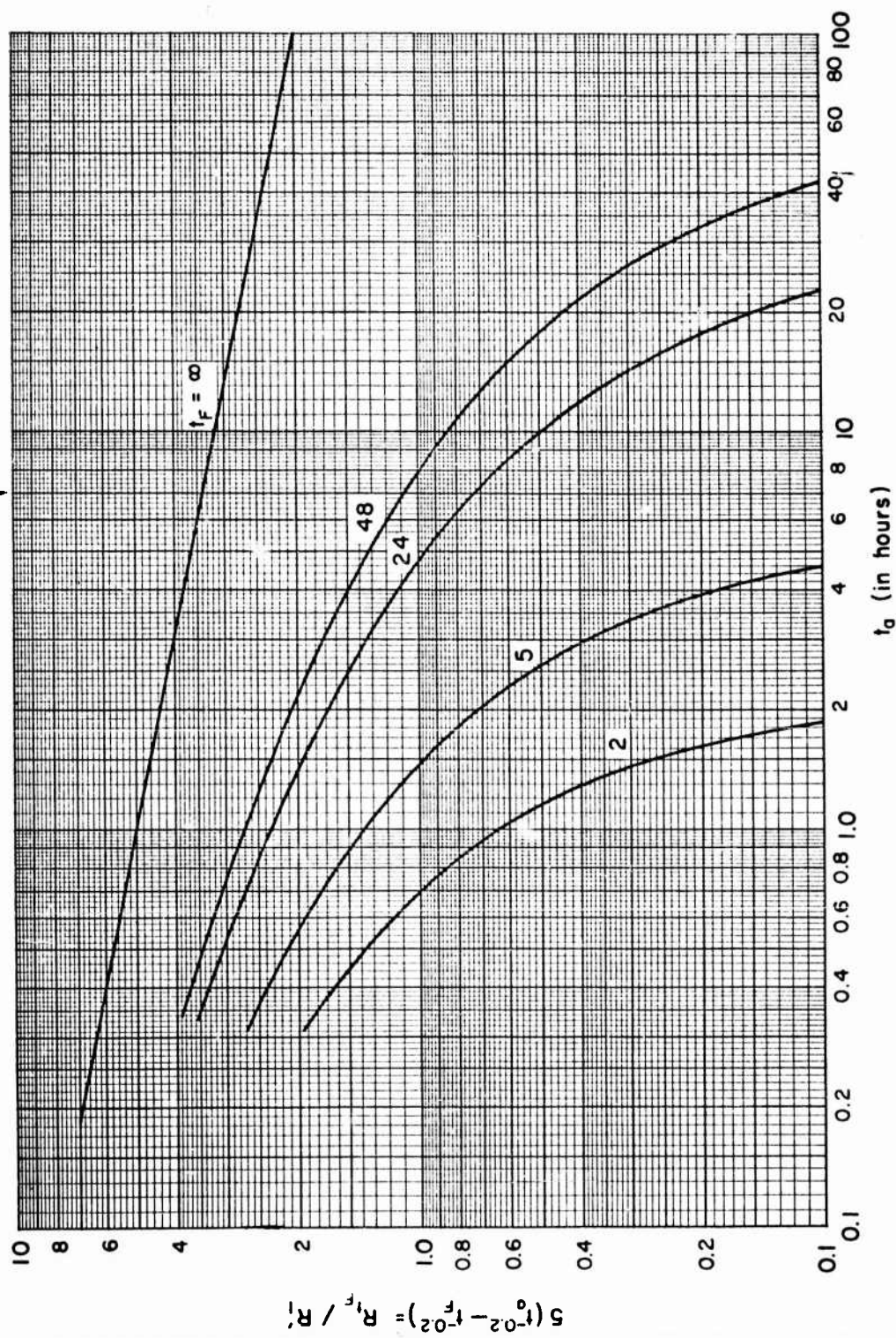
Fig.13 — Ratio of integrated dose to 1 hr dose rate as a function of near time of arrival for several final times
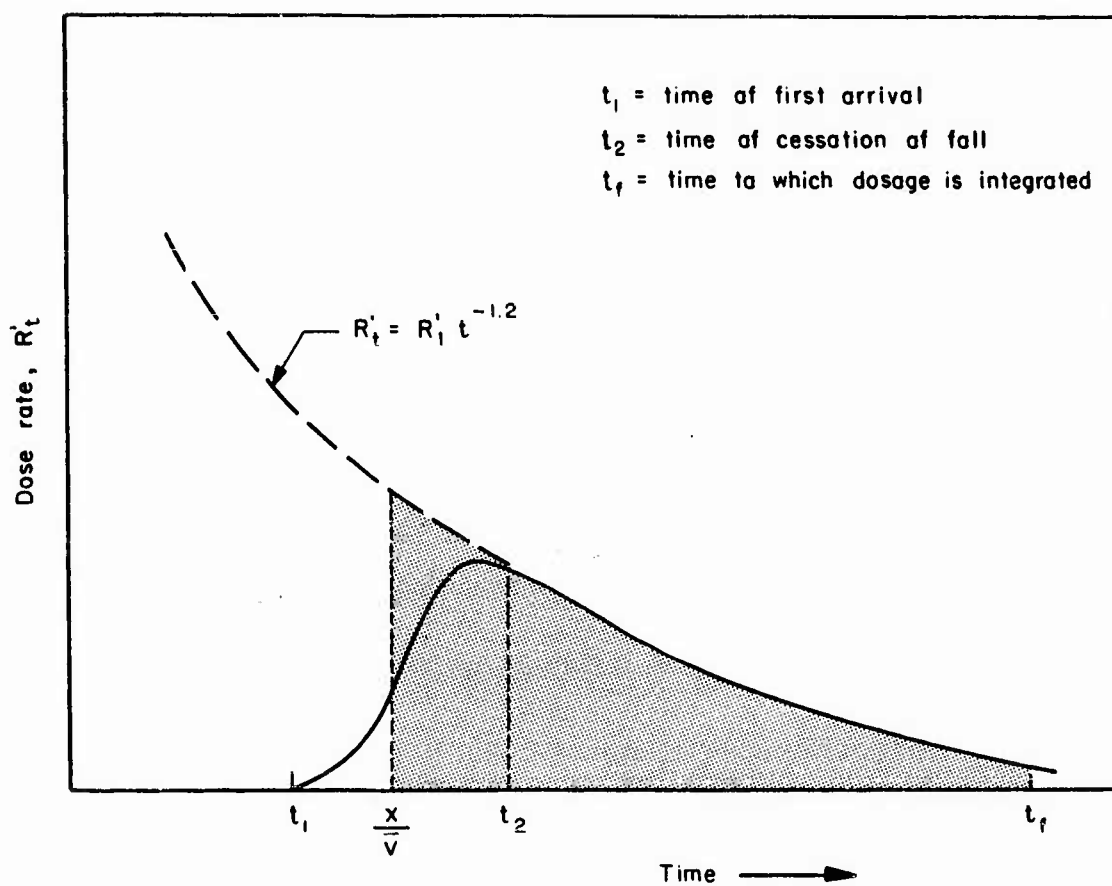
Fig. 14 — Estimated integrated dose compared
with true integrated dose

VII. CONCLUSIONS

The two solutions of the mathematical fallout model provide a basis for the construction of simple and rapid computation of fallout contours. A study now under way indicates that a slide rule may be constructed to solve Eq. (40). Another study indicates that wind shear and the vertical distribution with height can be taken into account by applying the model to layers of the cloud.

The computational procedures outlined here provide a fairly precise solution to the proposed mathematical model. Computational procedures do not introduce errors of more than 5 to 10 per cent. The degree to which the model reproduces the natural occurrence may be wrong by factors of 2 to 4. Improvement in the model and the values of the parameters in the model must wait for more and better observations of the phenomena of fallout.